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A Fifth-order Relaxation Scheme for the Shallow Water Equations*

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Abstract: A fifth-order relaxation scheme for the shallow water equations is presented in this paper. The scheme is based upon the fifth-order weighted essentially nonoscillatory (WENO) reconstruction and the implicit-explicit Runge-Kutta scheme. The resulting scheme does not require Riemann solvers and the computation of Jacobians, so it has the advantages of relaxation schemes. The resulting method is applied to simulate the one-dimensional dam-break problems on flat and non-flat topography. The results demonstrate the robustness and effectiveness of the present method. The effect of bottom friction is also discussed.

Keywords: shallow water equations; relaxation scheme; WENO reconstruction

Classification: AMS(2000) 65M06; 76M20 **CLC number:** O241.82 **Document code:** A

1 Introduction

The shallow water equations arise in many different applications, such as dam-break flows, prediction of water pollution, hydraulic jump, tidal flows in estuary, flood waves, etc. Recently, high-resolution relaxation schemes originally suggested in [1] have attracted much attention. In comparison with upwind schemes such as the Godunov scheme, relaxation schemes do not require the it Riemann solvers and the computation of Jacobians. These features make the relaxation schemes particularly suitable for those systems where the Riemann problem is difficult to solve or when it is not possible to perform analytical expression for Jacobians. The relaxation schemes were successfully implemented to the incompressible Euler equations and Hamilton-Jacobi equations.

As for application and implementation in solving the shallow water equations, Delis and Katsaounis[1] had adopted the first-order and second-order relaxation schemes introduced in [2]. Recently, Seaïd[3] presented a general framework to generalize the relaxation schemes to higher orders of accuracy and developed a third-order scheme for the shallow water equations. A fourth-order relaxation scheme for hyperbolic conservation laws can be found in [4]. In this paper we present a fifth-order relaxation scheme for approximating solutions of one-dimensional shallow water equations. Our method is a high-order extension of the scheme in [1,3,4]. In application, the present scheme is used to simulate dam-break problems.

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2 Governing equation

The one-dimensional shallow water equations can be written in conservation form as

$$\mathbf{U}_t + \mathbf{F}(\mathbf{U})_x = \mathbf{S}(\mathbf{U}) \,, \tag{1}$$

where

$$\mathbf{U} = \begin{bmatrix} h \\ hu \end{bmatrix}, \quad \mathbf{F}(\mathbf{U}) = \begin{bmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 \end{bmatrix}, \quad \mathbf{S}(\mathbf{U}) = \begin{bmatrix} 0 \\ -ghB_x - ghS_{fx} \end{bmatrix}.$$

Here, h is the water depth, u is the velocity, $B(x)$ is the bottom topography, g is the gravitational constant and $S_{fx} = n^2 u|u|h^{-\frac{4}{3}}$ denotes the friction slope, where n is the roughness coefficient.

3 A fifth-order relaxation scheme

Based on the ideas presented in [1,2], the one-dimensional shallow water equations (1) can be replaced by the following relaxation system

$$\begin{aligned} &\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{V}}{\partial x} = \mathbf{S}(\mathbf{U}), \\ &\frac{\partial \mathbf{V}}{\partial t} + \mathbf{A}\frac{\partial \mathbf{U}}{\partial x} = -\frac{1}{\tau}(\mathbf{V} - \mathbf{F}(\mathbf{U})), \end{aligned} \tag{2}$$

where $\mathbf{V} \in \mathbf{R}^2$, $\tau > 0$ is the relaxation rate and $\mathbf{A} = \mathrm{diag}\{a_1, a_2\}$ is a positive diagonal matrix. Under certain conditions the solution of (2) converges to the solution of the original problem (1) as $\tau \to 0$. Without loss of generality, let us consider the following uniform spatial grid

$$x_j = j\Delta x, \quad x_{j\pm\frac{1}{2}} = \Big(j \pm \frac{1}{2}\Big)\Delta x.$$

Introduce cell average of the variable $\mathbf{U}$ in $I_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$ as

$$\mathbf{U}_j(t) = \frac{1}{\Delta x}\int_{I_j} \mathbf{U}(x,t)\,\mathrm{d}x.$$

The approximate point value of $\mathbf{U}$ at $(x_{j+\frac{1}{2}}, t)$ is denoted by $\mathbf{U}_{j+\frac{1}{2}}(t)$. A spatial discretization to (2) in conservation form is given by

$$\begin{aligned} &\frac{\partial \mathbf{U}_j}{\partial t} + \frac{1}{\Delta x}\big(\mathbf{V}_{j+\frac{1}{2}} - \mathbf{V}_{j-\frac{1}{2}}\big) = \mathbf{S}(\mathbf{U})_j, \\ &\frac{\partial \mathbf{V}_j}{\partial t} + \frac{1}{\Delta x}\mathbf{A}\big(\mathbf{U}_{j+\frac{1}{2}} - \mathbf{U}_{j-\frac{1}{2}}\big) = -\frac{1}{\tau}\big(\mathbf{V}_j - \mathbf{F}(\mathbf{U})_j\big). \end{aligned} \tag{3}$$

Assuming that the cell-averages $\{\mathbf{U}_j\}$ are known, our goal is to compute the cell point values $\mathbf{U}_{j+\frac{1}{2}}$ and $\mathbf{V}_{j+\frac{1}{2}}$, the flux $\mathbf{F}(\mathbf{U})_j$ and the source $\mathbf{S}(\mathbf{U})_j$. The computation proceeds in two steps. As in [3], the first step is to perform a reconstruction from the given cell-averages

$$\tilde{\mathbf{U}}(x) = \sum_j \mathbf{R}_j(x;\mathbf{U})\chi_j(x), \tag{4}$$

where χ_j is the characteristic function of the interval I_j and $\mathbf{R}_j(x;\mathbf{U})$ is a polynomial defined in I_j. Given such a reconstruction, the point-values of $\tilde{\mathbf{U}}$ at the interface points are denoted by

$$\mathbf{U}^-_{j+\frac{1}{2}} = \mathbf{R}_j(x_{j+\frac{1}{2}};\mathbf{U}) \quad \text{and} \quad \mathbf{U}^+_{j-\frac{1}{2}} = \mathbf{R}_j(x_{j-\frac{1}{2}};\mathbf{U}).$$

To improve resolution and reduce numerical dissipation, the fifth-order WENO reconstruction[5] is utilized in this paper. With this background the computation of $\mathbf{U}^{\pm}_{j\mp\frac{1}{2}}$ from fifth-order WENO reconstruction is

$$U^-_{j+\frac{1}{2}} = R_j\big(x_{j+\frac{1}{2}};\mathbf{U}\big) = \sum_{r=0}^{2}\omega_r q^{(r)}_{j+\frac{1}{2}}, \quad U^+_{j-\frac{1}{2}} = R_j\big(x_{j-\frac{1}{2}};\mathbf{U}\big) = \sum_{r=0}^{2}\tilde{\omega}_r \tilde{q}^{(r)}_{j-\frac{1}{2}},$$

where $U_{j+\frac{1}{2}}$ and $V_{j+\frac{1}{2}}$ are the k-th components of $\mathbf{U}_{j+\frac{1}{2}}$ and $\mathbf{V}_{j+\frac{1}{2}}$, respectively. The $q^{(r)}_{j+\frac{1}{2}}$ and $\tilde{q}^{(r)}_{j-\frac{1}{2}}$ are

$$\begin{aligned}
q^{(0)}_{j+\frac{1}{2}} &= \frac{1}{3}U_j + \frac{5}{6}U_{j+1} - \frac{1}{6}U_{j+2}, & q^{(1)}_{j+\frac{1}{2}} &= -\frac{1}{6}U_{j-1} + \frac{5}{6}U_j + \frac{1}{3}U_{j+1},\\
q^{(2)}_{j+\frac{1}{2}} &= \frac{1}{3}U_{j-2} - \frac{7}{6}U_{j-1} + \frac{11}{6}U_j, & \tilde{q}^{(0)}_{j-\frac{1}{2}} &= \frac{11}{6}U_j - \frac{7}{6}U_{j+1} + \frac{1}{3}U_{j+2},\\
\tilde{q}^{(1)}_{j-\frac{1}{2}} &= \frac{1}{3}U_{j-1} + \frac{5}{6}U_j - \frac{1}{6}U_{j+1}, & \tilde{q}^{(2)}_{j-\frac{1}{2}} &= -\frac{1}{6}U_{j-2} + \frac{5}{6}U_{j-1} + \frac{1}{3}U_j.
\end{aligned}$$

The weights ω_r and $\tilde{\omega}_r$ are given by

$$\omega_i = \frac{\alpha_i}{\sum_{m=0}^{2}\alpha_m}, \quad \alpha_i = \frac{d_i}{(\epsilon+\beta_i)^2}, \quad \tilde{\omega}_i = \frac{\tilde{\alpha}_i}{\sum_{m=0}^{2}\tilde{\alpha}_m}, \quad \tilde{\alpha}_i = \frac{\tilde{d}_i}{(\epsilon+\beta_i)^2}, \quad i = 0,1,2.$$

Here $d_0 = \tilde{d}_2 = \frac{3}{10}$, $d_1 = \tilde{d}_1 = \frac{3}{5}$, $d_2 = \tilde{d}_0 = \frac{1}{10}$ and $\epsilon = 10^{-6}$. The smoothness indicators, β_i, are calculated by

$$\begin{aligned}
\beta_0 &= \frac{13}{12}\big(U_j - 2U_{j+1} + U_{j+2}\big)^2 + \frac{1}{4}\big(3U_j - 4U_{j+1} + U_{j+2}\big)^2,\\
\beta_1 &= \frac{13}{12}\big(U_{j-1} - 2U_j + U_{j+1}\big)^2 + \frac{1}{4}\big(U_{j-1} - U_{j+1}\big)^2,\\
\beta_2 &= \frac{13}{12}\big(U_{j-2} - 2U_{j-1} + U_j\big)^2 + \frac{1}{4}\big(U_{j-2} - 4U_{j-1} + 3U_j\big)^2.
\end{aligned}$$

Then, the k-th components of variables $\mathbf{V} \pm \mathbf{A}^{\frac{1}{2}}\mathbf{U}$ is discretized by

$$\big(\mathbf{V} + \sqrt{a_k}\mathbf{U}\big)_{j+\frac{1}{2}} = R_j\big(x_{j+\frac{1}{2}};\mathbf{V} + \sqrt{a_k}\mathbf{U}\big), \quad \big(\mathbf{V} - \sqrt{a_k}\mathbf{U}\big)_{j+\frac{1}{2}} = R_{j+1}\big(x_{j+\frac{1}{2}};\mathbf{V} - \sqrt{a_k}\mathbf{U}\big),$$

where V is the k-th components of $\mathbf{V}$. The point values are obtained by

$$U_{j+\frac{1}{2}} = \frac{1}{2\sqrt{a_k}}\big(R_j(x_{j+\frac{1}{2}};\mathbf{V} + \sqrt{a_k}\mathbf{U}) - R_{j+1}(x_{j+\frac{1}{2}};\mathbf{V} - \sqrt{a_k}\mathbf{U})\big),$$

$$V_{j+\frac{1}{2}} = \frac{1}{2}\big(R_j(x_{j+\frac{1}{2}};\mathbf{V} + \sqrt{a_k}\mathbf{U}) + R_{j+1}(x_{j+\frac{1}{2}};\mathbf{V} - \sqrt{a_k}\mathbf{U})\big).$$

Here $U_{j+\frac{1}{2}}$ and $V_{j+\frac{1}{2}}$ are the k-th components of $\mathbf{U}_{j+\frac{1}{2}}$ and $\mathbf{V}_{j+\frac{1}{2}}$, respectively. The next step is to approximate the flux $\mathbf{F}(\mathbf{U})_j$ and the source $\mathbf{S}(\mathbf{U})_j$ by the Simpson's quadrature.

To implement the time discretization, the formulation (3) is rewritten in common ordinary differential equations form

$$\frac{\mathrm{d}Y}{\mathrm{d}t} = L(Y) - \frac{1}{\tau}\tilde{L}(Y), \tag{5}$$

where

$$Y = \begin{pmatrix} \mathbf{U}_i \\ \mathbf{V}_i \end{pmatrix}, \quad L(Y) = \begin{pmatrix} \mathbf{S}(\mathbf{U})_i - D_x\mathbf{V}_i \\ -\mathbf{A}D_x\mathbf{V}_i \end{pmatrix}, \quad \tilde{L}(Y) = \begin{pmatrix} 0 \\ \mathbf{V}_i - \mathbf{F}(\mathbf{U})_i \end{pmatrix}.$$

Here

$$D_x\mathbf{V}_i = \frac{1}{\Delta x}(\mathbf{V}_{j+\frac{1}{2}} - \mathbf{V}_{j-\frac{1}{2}}).$$

The third-order implicit-explicit Runge-Kutta scheme proposed in [6] is utilized in this paper. When applied to system (5) it can be written as

$$K_l = Y^n + \Delta t\sum_{m=1}^{l-1}\bar{c}_{lm}L(K_m) - \frac{\Delta t}{\tau}\sum_{m=1}^{3}c_{lm}\tilde{L}(K_m), \quad l = 1,2,3,$$

$$Y^{n+1} = Y^n + \Delta t\sum_{l=1}^{3}\bar{d}_l L(K_l) - \frac{\Delta t}{\tau}\sum_{l=1}^{3}d_l\tilde{L}(K_l).$$

The 3×3 matrices $\bar{C} = (\bar{c}_{lm})$ and $C = (c_{lm})$ are given by

$$\bar{C} = \begin{pmatrix} 0 & 0 & 0 \\ \gamma & 0 & 0 \\ \gamma-1 & 2-2\gamma & 0 \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \gamma & 0 \\ 0 & 2-2\gamma & \gamma \end{pmatrix},$$

where $\gamma = \frac{3+\sqrt{3}}{6}$. The coefficient vectors $\bar{d} = (\bar{d}_1, \bar{d}_2, \bar{d}_3)^T$ and $d = (d_1, d_2, d_3)^T$ are given as

$$\bar{d} = \left(0, \frac{1}{2}, \frac{1}{2}\right)^T, \quad d = \left(0, \frac{1}{2}, \frac{1}{2}\right)^T.$$

4 Numerical examples

In this section, several test problems are presented. The results demonstrate the performance of the proposed scheme. In all the numerical examples below, the computational parameters are $\tau = 10^{-5}$, $\sqrt{a_1} = \sup|u - \sqrt{gh}|$ and $\sqrt{a_2} = \sup|u + \sqrt{gh}|$.

Example 1: One-dimensional dam-break problem

This problem is used to test the accuracy of the algorithm. Consider a horizontal rectangular channel with 2000m long. A dam is located at 1000m. The initial upstream and downstream water depths, h_u and h_d, are given by following two cases: 1) $h_u = 10m, h_d = 5m$; 2) $h_u = 10m, h_d = 0.5m$. Thus the water depth ratios h_d/h_u are 0.5 and 0.05. The computational domain is discretized into 200 cells. The bed slope is set to zero. Result at $t = 50s$ for depth ratio 0.5 is shown in Figure 1. It can be seen that the shock and rarefaction waves are well resolved by the present method. Figure 2 shows the result at $t = 50s$ for depth ratio 0.05. In this case, the roughness coefficient is considered and set as 0, 0.02 and 0.04, respectively.

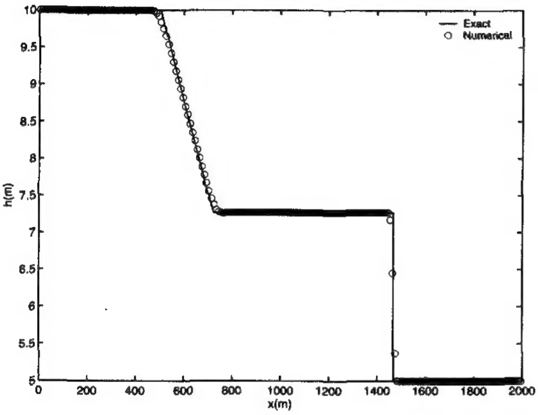

Figure 1: Water height, $h_d/h_u = 0.5$

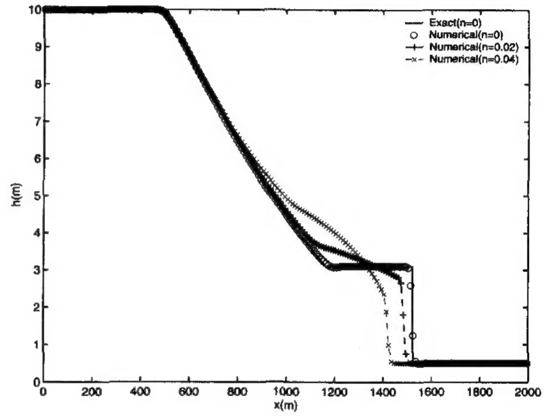

Figure 2: Water height, $h_d/h_u = 0.05$

Example 2: Dam-break problem over a continuous riverbed

We verify our scheme on the dam-break problem over a continuous riverbed. This test case is taken from [7]. The bottom function, $B(x) = 1.398 - 0.347\tanh(8x-4)$. The computaitonal domain is [0,1]. The initial conditions are

$$h(x,0) = \begin{cases} 1.0 & x \le 0.6, \\ 0.2 & x > 0.6, \end{cases} \qquad hu(x,0) = 0.0.$$

The gravitational constant is set to $g = 1.0$. A uniform grid with 100 grid points is chose in this case, and the numerical solution is plotted together with the reference solution which is computed by the present scheme with 3000 grid points. Figure 3 shows the result at $t = 0.25$. The simulated result demonstrates the effectiveness of our method.

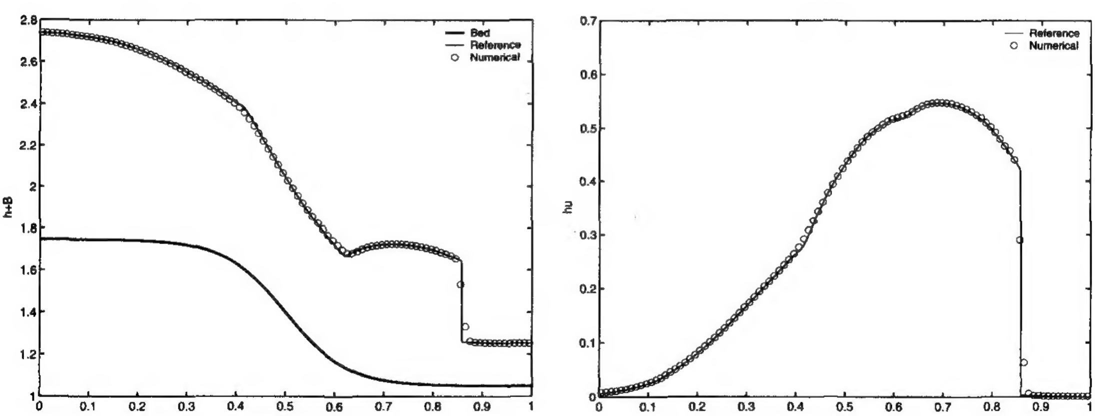

Figure 3: Water height and discharge plot for dam-break problem over a continuous riverbed

References:

[1] Delis A I, Katsaounis T. Relaxation schemes for the shallow water equations[J]. International Journal for Numerical Methods in Fluids, 2003, 41(7): 695-719
[2] Jin S, Xin Z. The relaxation schemes for systems of conservation laws in arbitrary space dimensions[J]. Communications on Pure Applied Mathematics, 1995, 48(3): 235-276
[3] Seaïd M. Non-oscillatory relaxation methods for the shallow water equations in one and two space dimensions[J]. Internation Journal for Numerical Methods in Fluids, 2004, 46(5): 1-28
[4] Chen J Z, Shi Z K. Application of a fourth-order relaxation scheme to hyperbolic systems of conservation laws[J]. Acta Mechanica Sinica, 2006, 22(1): 84-92
[5] Jiang G S, Shu C W. Efficient implementation of weighted ENO cchemes[J]. Journal of Computational Physics, 1996, 126(1): 202-228
[6] Pareschi L, Russo G. Implicit-explicit Runge-Kutta schemes and applications to hyperbolic systems with relaxation[J]. Journal of Scientific Computing, 2005, 25(1): 129-155
[7] Jin S. A steady-state capturing method for hyperbolic systems with geometrical source terms[J]. Mathematical Modelling and Numerical Analysis, 2001, 35(4): 631-646

求解浅水方程的五阶松弛格式

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摘　要: 对浅水方程，提出了一种具有五阶精度的松弛格式。该格式以五阶WENO重构和隐式Runge-Kutta方法为基础。格式保持了松弛格式简单的优点，即不用求解Riemann问题和计算通量函数的雅可比矩阵。应用该方法对一维平底和非平底溃坝问题进行了数值模拟，结果表明方法健全、有效。对摩阻源项也进行了讨论。

关键词: 浅水方程；松弛格式；WENO重构